Laminar Equilibrium in two-resource Resource Selection Games*

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Abstract

A variety of real-life scenarios can be modeled as a resource selection game (RSG) with two resources: n agents are to utilize from one of two available resources, $i = 1, 2$. When resource i is selected by q agents each incurs a congestion cost equal to $f_i(q)$, where the function f_i is strictly increasing in q. Agents strive to minimize the congestion costs they incur. In this setting it is known that an equilibrium may not exist if no restrictions are imposed on coalition formation. Motivated by real-life hierarchic communities we study equilibria under a natural restriction on coalition formation—when the set of viable coalitions exhibits a laminar structure. Our main result is that under the restriction of laminarity an equilibrium allocation always exists in a two-resource RSG.

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1 Introduction

A class of games that has drawn attention in the literature is the Resource Selection Games (RSGs): There is a set of agents, N, who are to utilize from a set of resources, M. Each agent selects one of the resources, resulting in an allocation $a = (a_i)_{i=1}^{|M|}$, where $a_i \subseteq N$ is the set of agents utilizing from resource i. An agent that utilizes from resource i incurs a congestion cost $f_i(|a_i|)$, where the congestion cost function f_i is strictly increasing in $|a_i|$. Agents strive to minimize their incurred congestion costs by optimally choosing the resources from which they utilize. RSGs fall into the class of congestion games and can be used to model real-life scenarios in a wide range of areas including economics, transportation, communication networks, and biology.[1]

In this paper we study a restricted form of RSGs: We consider RSGs with two resources (i.e., $|M| = 2$). This specific form of the game arises interest on practical grounds, as it can be used in the analysis of a variety of real-life scenarios such as:

- In the city of Istanbul, vehicles ("agents") need to use one of two bridges ("resources") in order to travel between the Asian and the European parts of the city. The congestion cost incurred by a vehicle using a bridge (includes the costs of fuel and time) increases as the number of vehicles using that bridge increases.

- Boeing and Airbus ("resources") are two jet manufacturers producing aircrafts. Customers ("agents") who wish to buy an aircraft from one of these manufacturers are charged more (i.e., the "congestion cost" increases) if the demand for aircrafts from that manufacturer increases.

- At a college a course is offered to students ("agents") in two sections ("resources"). For a better learning experience students strive to avoid registering the crowded section (where the "congestion cost" is higher).

The purpose of our paper is to investigate the existence of "equilibrium" allocations in the two-resource setting. In non-cooperative games, equilibrium notions are defined on the basis of the absence of "profitable deviations" by coalitions of agents. In our two-resource setting we define a profitable deviation as follows: Let $c \subseteq N$ be a coalition of agents. A *deviation* by coalition c is a sequence (c_1, c_2) such that $c_1 \cap c_2 = \emptyset$ and $c_1 \cup c_2 = c$. Here, c_1 and c_2 are the sets of coalition members that deviate to resource 1 and resource 2, respectively. In words, a deviation is an agreement by coalition members on how they are to utilize from the two

[1] The class of congestion games was introduced by Rosenthal [14] and shown to coincide (up to an isomorphism) with the class of finite potential games by Monderer and Shapley [12]. For other related studies on congestion games see [5, 8, 9, 11]. For applications of RSGs, see [10, 13].

resources. If coalition c takes deviation (c_1, c_2) at allocation a, this induces a new allocation. We say that (c_1, c_2) is a *profitable deviation* at a if at the induced allocation each coalition member becomes weakly better off and at least one coalition member becomes better off.

In non-cooperative games a variety of equilibrium notions can be defined by imposing various restrictions on coalition formation. Two widely-studied equilibrium notions are a "super strong equilibrium" and a "Nash equilibrium." An allocation is a *super strong equilibrium* if no coalition has a profitable deviation (i.e., coalition formation is unrestricted). An allocation is a *Nash equilibrium* if no agent has a profitable deviation (i.e., coalition formation is restricted to singletons).[2] Clearly a super strong equilibrium is a Nash equilibrium but not vice versa. RSGs fall into the class of potential games for which the existence of a Nash equilibrium is guaranteed (see [12, 14]). Alas, the following example, due to Feldman and Tennenholtz [4], illustrates that a super strong equilibrium may not exist in a RSG with two resources.[3] (Example 2 in Section 3 generalizes this impossibility result to the m-resource case, where $m \geq 2$ and not all resources are identical.)

EXAMPLE 1 *Consider the RSG where* $N = \{1, 2, 3\}$, $M = \{1, 2\}$, *and* $f_i(q_i) = q_i$ *for* $i = 1, 2$.

In this RSG there exists no super strong equilibrium. To see this note that: At an allocation where all agents are assigned to the same resource, an agent that deviates to the other resource becomes better off. In all other allocations, two agents are assigned to one of the resources and one agent is assigned to the other resource. Wlog, let agents 1 and 2 be assigned to resource 1 and agent 3 to resource 2. But now the coalition $c = \{1, 2\}$ *has a profitable deviation: When agent 1 deviates to resource 2 and agent 2 remains at resource 1, agent 1 becomes weakly better off and agent 2 becomes better off.* $\diamondsuit$

The above example is what spurred the theoretical analysis in this paper. When coalition formation is unrestricted since an equilibrium allocation may not exist, we study equilibria under a natural restriction on coalition formation. The restriction that we consider is motivated by "hierarchically" organized communities that are widespread in real life. For instance, a military is divided into corps, legions, and brigades; a cabinet is divided into ministries, departments, and directorates; a university is divided into faculties and departments; and a company is divided into business units, divisions, and departments. If we take the company example, for

[2]This paper deals with only pure strategies, and hence, all equilibrium notions studied are pure-strategy equilibrium notions.

[3]Our paper defines a profitable deviation based upon the "weak domination" relation: the deviation makes at least one coalition member better off. An alternative definition would be based upon the "strong domination" relation: the deviation makes every coalition member better off. Using this latter approach, Aumann [3] introduced the notion of a "strong equilibrium." In the context of (monotonic) congestion games, Holzman and Law-Yone [7] showed that the set of strong equilibria is non-empty and indeed coincides with the set of Nash equilibria.

instance, note that the company's each structural unit is headed by a manager (or a management team) who has the ability and/or authority to coordinate the actions taken by the staff under that structural unit. Often regulations or organizational difficulties render coordinated action inviable for "mixed structural units," however. For instance, the employees in two separate departments under two separate divisions may find it hard to communicate with one another; or, even if they could communicate, company regulations may prevent them from taking coordinated action. Then, when studying how the company's resources are utilized, it may be reasonable to assume that only a whole structural segment of the company, with the supervision and authority of its manager (or management team), can take coordinated action.

We formalize our above discussion as follows. Let $\mathcal{P}_{\geq 1}(N) : \{c | c \subseteq N,\ c \neq \emptyset\}$. Let $\mathcal{P}_{=1}(N) : \{c | c \subseteq N,\ |c| = 1\}$. A set $C \subseteq \mathcal{P}_{\geq 1}(N)$ is called a *coalition structure*. An allocation a is called C*-stable* if at a no coalition in C has a profitable deviation. That is, a coalition structure C is a restriction on coalition formation; and a C-stable allocation is an equilibrium allocation when coalition formation is restricted to the set C. Expressed in our terms, a super strong equilibrium is a $\mathcal{P}_{\geq 1}(N)$-stable allocation and a Nash equilibrium is a $\mathcal{P}_{=1}(N)$-stable allocation. Motivated by real-life hierarchic communities we consider "laminar" coalition structures: A coalition structure C is *laminar* if for any $c_1, c_2 \in C$ such that $c_1 \cap c_2 \neq \emptyset$, either $c_1 \subseteq c_2$ or $c_2 \subseteq c_1$. For instance, in our company example above note that if a department and a business unit are not disjoint, then the department must be under that business unit.

Clearly $\mathcal{P}_{=1}(N)$ is laminar and $\mathcal{P}_{\geq 1}(N)$ is not. The fact that a $\mathcal{P}_{=1}(N)$-stable allocation always exists but a $\mathcal{P}_{\geq 1}(N)$-stable allocation may not leads to the natural question as to whether or not there is a relationship between laminarity and the existence of an equilibrium allocation. The main result of our paper gives an affirmative answer to this question in the two-resource case: In Theorem 2, we show that in a two-resource RSG, for C laminar, there always exists a C-stable allocation (which we refer to as a *laminar equilibrium*). One corollary of this finding is that in the two-resource case, for C laminar, there always exists an allocation which is both C-stable and a Nash equilibrium (Corollary 2).

To our best knowledge, our paper is the first study in the literature where the restriction of laminarity is imposed on coalition formation in a game-theoretic setting. In a related study, Feldman and Tennenholtz [4] considered the notion of a "partition equilibrium," in which coalition formation is presumed to be restricted to a partition of the set of agents. Obviously a partition equilibrium is a special instance of the more general notion of a laminar equilibrium. Anshelevich et al. [1] showed that in a RSG there always exists a partition equilibrium.[4]

[4]For two other related studies, see [2 and 6].

Note that our Theorem 2 generalize their result in the two-resource setting by showing that a laminar equilibrium always exists. Since in real life hierarchic communities are widespread, we believe future studies on laminarity may become worthwhile contributions to the game-theoretic literature—in particular, in those games where a super strong equilibrium may not exist. In the context of RSGs, for instance, one open question is whether or not a laminar equilibrium always exists when the number of resources is not necessarily two. Although our paper does not resolve this more general question, our technical contributions may perhaps become useful in studying this open question in future studies.

The rest of the paper is organized as follows: Section 2 introduces our model. Section 3 discusses the notions of a super strong equilibrium and a Nash equilibrium. Section 4 introduces laminar equilibrium and presents the two-color theorem of laminarity, which is at the heart of our proof of Theorem 2. Section 5 presents Theorem 2, the main result of our paper.

2 MODEL

A *resource selection game* (RSG) is a triplet $\langle N, M, f \rangle$ where:

- $N : \{1, 2, \cdots, n\}$ is the set of *agents*;
- $M : \{1, 2, \cdots, m\}$ is the set of *resources*;
- $f : (f_i)_{i=1}^m$ is the profile of strictly monotonic (congestion) *cost functions*.

When q agents utilize from resource i, each incurs a cost equal to $f_i(q)$. Each agent strives to minimize the cost that it incurs. In the rest of the paper we fix the game $\langle N, M, f \rangle$.[5]

An *allocation* is a sequence $a : (a_i)_{i=1}^m$ such that:

- for each $i \in M$, $a_i \subseteq N$;
- for every $i, i' \in M$ such that $i \neq i'$, $a_i \cap a_{i'} = \emptyset$;
- $\bigcup_{i \in M} a_i = N$.

Above, a_i denotes the set of agents that are assigned to resource i at allocation a. Thus, at a each agent in a_i incurs a cost equal to $f_i(|a_i|)$. Let $\mathcal{A}$ be the domain of allocations.

[5]Throughout, we use $\subset$ and $\subseteq$ to denote the "strict subset of" and the "subset of" relations. For a number x and a set X, we use $|x|$ and $|X|$ to denote the "absolute value of x" and "the cardinality of X," respectively.

The *maxcost* of an allocation a is the cost at the resource whose cost is highest. That is, the maxcost of allocation a equals $max_{i \in M}\ f_i(|a_i|)$.

The *minmaxcost* of the RSG, to be denoted by u, is the maxcost of the allocation whose maxcost is smallest. That is, $u = min_{a \in \mathcal{A}}\ max_{i \in M}\ f_i(|a_i|)$.

A *minmaxcost allocation* is an allocation whose maxcost equals u. That is, a is a minmax-cost allocation if $max_{i \in M}\ f_i(|a_i|) = u$.

Let $q_i = max_{q \in \mathbb{Z}_{\geq 0}}\ f_i(q) \leq u$. We refer to q_i as resource i's quota. That is, a resource's quota is the maximum number of agents that it can be assigned without making its cost exceed u. We distinguish between resources whose cost may and may not become equal to u. A resource i is a:

- *type 1 resource* if $f_i(q_i) = u$;
- *type 2 resource* if $f_i(q_i) < u$.

Let T_1 and T_2 denote, respectively, the sets of type 1 and type 2 resources. Since the minmaxcost of the game is u, we have $T_1 \neq \emptyset$. Also, for $i \in T_1$, let $\beta_i = f_i(q_i - 1)$. That is, for a type 1 resource i, β_i is its cost when the number of agents assigned to it is one less than its quota.

We say that agent j is *weakly better off* (is *better off*) at allocation a than at a' if for $i, i' \in M$ such that $j \in a_i$ and $j \in a'_{i'}$, we have $f_i(|a_i|) \leq f_{i'}(|a'_{i'}|)$ (we have $f_i(|a_i|) < f_{i'}(|a'_{i'}|)$). That is, agent j is weakly better off (is better off) if the cost it incurs is weakly smaller (is smaller).

A *coalition* c is a non-empty subset of agents; i.e., $c \subseteq N$ and $c \neq \emptyset$. Let $\mathcal{P}_{\geq 1}(N)$ be the domain of coalitions; i.e., $\mathcal{P}_{\geq 1}(N) = \mathcal{P}(N) - \{\emptyset\}$, where $\mathcal{P}(N)$ is the power set of N.

A *coalition structure* C is a set of viable coalitions; i.e., $C \subseteq \mathcal{P}_{\geq 1}(N)$. For coalition structure $\mathcal{P}_{\geq 1}(N)$, any coalition is viable. Let $\mathcal{P}_{=1}(N) = \{c \subset N \mid |c| = 1\}$. For coalition structure $\mathcal{P}_{=1}(N)$, the viable coalitions are singletons.

2.1 Stability

Our theoretical analysis centers around the "stability" notion. Before defining stability we need some additional terminology.

A *deviation* by a coalition c is a sequence $(c_i)_{i=1}^{m}$ such that: $c_1 \cup c_2 \cup \cdots \cup c_m = c$, and for each $i, i' \in M$ and $i \neq i'$, $c_i \cap c_{i'} = \emptyset$. One should think of a deviation by a coalition as an agreement by coalition members on how they are to utilize from resources. The set c_i denotes the set of coalition members that agree to utilize from resource i.

When coalition c *takes deviation* $(c_i)_{i=1}^m$ at allocation a, this induces a new allocation, to be denoted by $a \circ (c_i)_{i=1}^m$. At $a \circ (c_i)_{i=1}^m$, the set of agents that are assigned to resource i is $(a_i \setminus c) \cup c_i$. That is, coalition members are reallocated to resources as specified by the deviation while the assignments of remaining agents do not change.

A deviation $(c_i)_{i=1}^m$ by coalition c is a *profitable deviation* at allocation a if at $a \circ (c_i)_{i=1}^m$, compared to at a, each agent in c is weakly better off and at least one of them is better off. That is, a profitable deviation leads to a Pareto-superior allocation for coalition c.

We now introduce our stability notions. An allocation a is:

- c-*stable* if coalition c does not have a profitable deviation at a;
- C-*stable* if for coalition structure C, a is c-*stable* for each $c \in C$.

3 Super Strong Equilibrium and Nash Equilibrium

Two particular stability notions that have been of interest in the literature are a "super strong equilibrium" and a "Nash equilibrium." Expressed in our terms, a *super strong equilibrium* is a $\mathcal{P}_{\geq 1}(N)$-stable allocation, and a *Nash equilibrium* is a $\mathcal{P}_{=1}(N)$-stable allocation.

A super strong equilibrium is clearly a very appealing notion of stability. Alas, in a RSG there may not exist a super strong equilibrium, as demonstrated in Example 1 in Section 1 in the two-resource case. In Example 2 below, we generalize this impossibility result to the m-resource case where $m \geq 2$ and not all resources are identical.

EXAMPLE 2 *Consider the RSG where* $N = \{1, 2, \cdots, m+1\}$, $M = \{1, 2, \cdots, m\}$, *and* $f_i(q_i) = q_i$ *for* $i = 1, 2$ *and* $f_i(q_i) = 2q_i - 0.5$ *for* $i = 3, 4, \cdots, m$. *We will show that in this game there exists no super strong equilibrium.*

By way of contradiction, suppose that there exists an allocation a *which is a super strong equilibrium.*

At a *suppose that for some resource* i, $a_i = \emptyset$. *Then there are* $m+1$ *agents assigned to the remaining* $m-1$ *resources. Thus, there exists a resource* $i' \neq i$ *such that* $|a_{i'}| \geq 2$. *Then* $f_{i'}(|a_{i'}|) \geq f_{i'}(2) \geq 2$. *For* $j \in a_{i'}$, *if agent* j *deviates from resource* i' *to* i, *the cost that* j

incurs becomes $f_i(1) \leq 1.5$. Thus, j becomes better off, contradicting that a is a super strong equilibrium. Thus, at a, for each resource i we must have $|a_i| \geq 1$. Since there are $m+1$ agents and m resources, this implies that there is a resource, say i, which is assigned two agents, while each remaining resource is assigned precisely one agent. Let $a_i = \{j, j'\}$.

If $i \geq 3$, the cost that j incurs at a is $f_i(2) = 3.5$. Then, if j deviates from resource i to resource 1, the cost that she incurs becomes $f_1(2) = 2$, and hence j becomes better off. This contradicts that a is a super strong equilibrium. Therefore, we must have $i \in \{1, 2\}$. Wlog, let $i = 1$. Note that at a if j deviates from resource 1 to resource 2, the cost incurred remains the same for j but decreases for j'. But then this is a profitable deviation for the coalition $\{j, j'\}$, contradicting that a is a super strong equilibrium.

Therefore, in this RSG there exists no super strong equilibrium. $\Diamond$

As it turns out, in a RSG there always exists a Nash equilibrium, however. Proposition 1 below characterizes the set of Nash equilibria in a RSG. It is due to Anshelevich et al [1] but for the sake of completeness we present a proof of this proposition below.

PROPOSITION 1 *In a RSG, the set of Nash equilibria is non-empty. And an allocation a is a Nash equilibrium (i.e., $\mathcal{P}_{=1}(N)-$stable) if and only if:*

(a) for each $i \in T_2$, $|a_i| = q_i$,

(b) for each $i \in T_1$, $|a_i| \in \{q_i - 1, q_i\}$,

(c) for some $i \in T_1$, $|a_i| = q_i$.

Proof. We first show the existence of an allocation a satisfying the conditions (a), (b), and (c) in the proposition. To do that, regarding the cardinality $|N|$ note that we must have:

$$|N| \leq \sum_{i \in M} q_i. \qquad (*)$$

Because otherwise it would be impossible to allocate agents to resources without the quota of one resource being exceeded, contradicting how we defined resource quotas. We must also have:

$$|N| \geq 1 + \sum_{i \in T_1} (q_i - 1) + \sum_{i \in T_2} q_i. \qquad (**)$$

Because otherwise it would be possible to allocate agents to resources such that each type 1 resource i is assigned $q_i - 1$ or fewer agents and each type 2 resource i is assigned

q_i or fewer agents, making the allocation's maxcost smaller than the game's minmaxcost, a contradiction.

Given the above inequalities, it is clear that there exists an allocation satisfying the conditions (a), (b), and (c).

We now show the biconditional statement.

(if)

By way of contradiction, suppose that allocation a is not a Nash equilibrium but it satisfies the conditions (a), (b), and (c). Then a triplet $(i, i', j) \in (M \times M \times N)$ exists such that $j \in a_i$, $i \neq i'$, and j becomes better off if it deviates at a from i to i'. Since $|a_i| \in \{q_i - 1, q_i\}$, at a the cost that j incurs is less than or equal to u. If $i' \in T_1$, then $|a_{i'}| \in \{q_{i'} - 1, q_{i'}\}$, and if $i' \in T_2$, then $|a_{i'}| = q_i$. In all cases, when j deviates to i', the cost incurred at i' becomes greater than or equal to u. But then the deviation does not make j better off, a contradiction. Therefore, any allocation that satisfies the conditions (a), (b), and (c) is a Nash equilibrium.

(only if)

By way of contradiction, suppose that allocation a is a Nash equilibrium but it does not satisfy one of the conditions (a), (b), and (c).

Suppose that for some resource i, either $i \in T_1$ and $|a_i| < q_i - 1$ or $i \in T_2$ and $|a_i| < q_i$. Then at a the cost incurred at resource i is less than u. Since the minmaxcost of the game is u, at a there exists a resource $i' \neq i$ such that at resource i' the cost incurred is greater than or equal to u. Note that $u > 0$ because $|N| > 0$. Then $a_{i'} \neq \emptyset$. Let $j \in a_{i'}$. At a if j deviates from resource i' to i, after the deviation the cost incurred at i becomes less than u. But this is then a profitable deviation for j, contradicting that a is a Nash equilibrium. Thus, it must be that for each $i \in T_1$, $|a_i| \geq q_i - 1$, and for each $i \in T_2$, $|a_i| \geq q_i$.

Suppose that there exists a resource, say i', such that $|a_{i'}| > q_{i'}$. Then $a_{i'} \neq \emptyset$ and at a the cost incurred at resource i' is greater than u. Let $j \in a_{i'}$. Note that if $|M| = 1$, we get $|N| = |a_{i'}| > q_{i'}$, contradicting that the minmaxcost of the game is u. Thus, $|M| \geq 2$. Consider a resource $i'' \neq i'$. If $|a_{i''}| < q_{i''}$, then j becomes better off when it deviates at a from i' to i'', contradicting that a is a Nash equilibrium. Thus, for each resource $i'' \neq i'$, $|a_{i''}| \geq q_{i''}$. Together with the fact that $|a_{i'}| > q_{i'}$, this implies that

$$|N| = \sum_{i \in M} |a_i| > \sum_{i \in M} q_i,$$

which contradicts $(*)$ above. Therefore, we obtain that for each resource i', $|a_{i'}| \leq q_{i'}$.

To sum up, above we have shown that: for each $i \in T_1$, $|a_i| \in \{q_i - 1, q_i\}$, and for each $i \in T_2$, $|a_i| = q_i$. These findings, together with $(**)$ above, imply that there exists a resource $i \in T_1$ such that $|a_i| = q_i$. But then the allocation a satisfies the conditions (a), (b), and (c), contradicting our initial supposition. Therefore, any allocation that is a Nash equilibrium satisfies the conditions (a), (b), and (c).

This concludes our proof. ■

A corollary of Proposition 1 is that if an allocation is a Nash equilibrium, then it is a minmaxcost allocation. The converse of this statement is not true, however, as shown in Example 3.

COROLLARY 1 *If allocation a is a Nash equilibrium then a is a minmaxcost allocation.*

EXAMPLE 3 *Let $N = \{1, 2, 3, 4, 5\}$ and $M = \{1, 2, 3\}$. Let $f_i(q) = 4q$ for $i = 1, 2$ and $f_3(q) = 3q$.*

It is clear that the minmaxcost of this game is 8. Consider the allocation a such that $a_1 = \{1, 2\}$, $a_2 = \{3, 4\}$, and $a_3 = \{5\}$. The maxcost of allocation a is 8 but a is not a Nash equilibrium: At a agent 1 becomes better off if it deviates from resource 1 to 3. ♢

We know from Examples 1 and 2 that in a RSG a super strong allocation may not exist. However, in particular cases of the RSG the set of super strong equilibria may be non-empty. Below in Proposition 2 we present a case where the set of super strong equilibria coincides with the set of Nash equilibria.[6]

PROPOSITION 2 *In a RSG suppose that $|N| = \sum_{i \in M} q_i$. Then an allocation a is a super strong equilibrium if and only if it is a Nash equilibrium.*

Proof. A super strong equilibrium is clearly a Nash equilibrium. To prove the lemma we need to show that the converse statement is also true.

Let $|N| = \sum_{i \in M} q_i$. By way of contradiction, suppose that allocation a is a Nash equilibrium but is not a super strong equilibrium. Since a is a Nash equilibrium, by Proposition 1 we know that for each $i \in M$, $|a_i| = q_i$. Since a is not a super strong equilibrium, there exists a profitable deviation $(c_i)_{i=1}^m$ by some coalition c at allocation a. Let $a' = a \circ (c_i)_{i=1}^m$.

Suppose that for some resource i, $|a'_i| > |a_i| = q_i$. Since $a'_i = a_i \setminus (a_i \cap c) \cup c_i$, we must have $|c_i| > |a_i \cap c|$. Then $c_i \neq \emptyset$. Let $j \in c_i$. Since $|a'_i| > q_i$, the cost that j incurs at a' is greater than u. But at a the cost that j incurs is less than or equal to u, contradicting that $(c_i)_{i=1}^m$ is a

[6]Proposition 2 also becomes useful later on in Section 5 to show Theorem 2, the main result of our paper.

profitable deviation at a. Thus, it must be that for each resource i, $|a'_i| \leq |a_i| = q_i$. Together with the fact that $|N| = \sum_{i \in M} q_i$, this implies that for each $i \in M$, $|a'_i| = q_i$. Then it is clear that the sum of the costs incurred by members of coalition c is the same at allocations a and a'. But then, if there exists an agent in c that is better off at a' (compared to at a) there must also exist an agent in c who is worse off at a', contradicting that $(c_i)_{i=1}^m$ is a profitable deviation at a. Therefore, a is a super strong equilibrium. ■

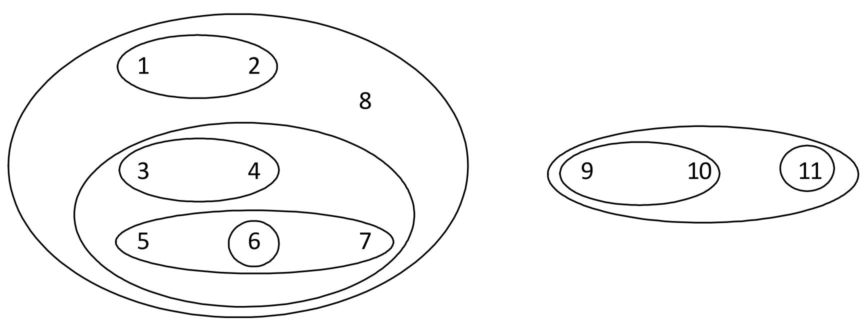
4 Laminar Coalition Structures and the Two-color Theorem of Laminarity

A coalition structure C is *laminar* if for any two coalitions $c_1, c_2 \in C$ such that $c_1 \cap c_2 \neq \emptyset$, either $c_1 \subseteq c_2$ or $c_2 \subseteq c_1$. The following example illustrates a laminar coalition structure.

EXAMPLE 4 *Let* $N : \{1, 2, \cdots, 11\}$*. The following is a laminar coalition structure:*

$$C = \left\{ \begin{array}{c} \{6\}, \{11\}, \{1,2\}, \{3,4\}, \{9,10\}, \\ \{5,6,7\}, \{9,10,11\}, \{3,4,5,6,7\}, \\ \{1,2,3,4,5,6,7,8\} \end{array} \right\}$$

To check that C is laminar see the diagram below: Each circle in the diagram indicates a coalition consisting of the enclosed agents.



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Note that $\mathcal{P}_{\geq 1}(N)$ is not laminar but $\mathcal{P}_{=1}(N)$ is. Another particular instance of a laminar coalition structure can be found in Anshelevich et al [1]: In the context of RSGs they consider the notion of a "partition equilibrium," where the presumption is that the set viable coalitions is a partition of the set of agents.

We present next what we call “the two-color theorem of laminarity.” It lies at the heart of the proof of Theorem 2, the main result of our paper. We believe our two-color theorem may also be of independent interest, in particular, in future studies on laminarity. In simple terms it states that for any $N' \subseteq N$ and any laminar coalition structure C, the set N' can be partitioned into two subsets of about equal size (i.e., into N'_B and $N'_W = N' \setminus N'_B$ where $||N'_B| - |N'_W|| \leq 1$) such that for each $c \in C$, the set of coalition members in N' (i.e., $N' \cap c$) also becomes partitioned into two subsets of about equal size (i.e., $||N'_B \cap c| - |N'_W \cap c|| \leq 1$). (The agents in N'_B and N'_W are referred to as “black agents” and “white agents,” respectively, and hence is the name of the theorem).

THEOREM 1 ***(the two-color theorem of laminarity)*** *Let $N' \subseteq N$, $|N'| \geq 1$. Let $k \geq 1$ be such that $|N'| = 2k-1$ or $|N'| = 2k$. Then, for any laminar coalition structure C, the set N' can be partitioned into the subsets N'_B and $N'_W = N' \setminus N'_B$ such that $|N'_B| = k$ and for each $c \in C$, $||N'_B \cap c| - |N'_W \cap c|| \leq 1$.*

Before proceeding with the proof of Theorem 1, we equip ourselves with some new terms and tools. Suppose that C is such that $N \in C$ and $\mathcal{P}_{=1}(N) \subseteq C$. For $c, c' \in C$, we say that “c is a child of c',” and “c' is the mother of c,” if $c \subset c'$ and there does not exist $c'' \in C$ such that $c \subset c'' \subset c'$. We recursively define the sets $C^1, C^2, \cdots, C^t$ as follows:

$$C^1 = \{c \in C \,|c \text{ is not a child of any coalition } c' \in C\},$$

and for $2 \leq s \leq t$,

$$C^s = \left\{c \in C \,\middle|c \text{ is a child of some coalition } c' \in C^{s-1}\right\}.$$

Above, t is set such that C^t is non-empty and no coalition in C^t has a child in C. Note that $C^1 = \{N\}$ and $C = C^1 \cup C^2 \cup \cdots \cup C^t$.

For $c \in C$, let $\#(c)$ be the number of children of c. For $\#(c) > 0$, we label the children of c as $c_1, c_2, \cdots, c_{\#(c)}$. To make precise our labeling we use the following rule: among coalitions $c_1, c_2, \cdots, c_{\#(c)}$, c_1 is the one that includes the smallest-index agent; among coalitions $c_2, \cdots, c_{\#(c)}$, c_2 is the one that includes the smallest-index agent; and so on. Note that, since $\mathcal{P}_{=1}(N) \subseteq C$, $c = c_1 \cup c_2 \cup \cdots \cup c_{\#(c)}$. We are now ready to proceed with our proof.

Proof. We can assume wlog that $N \in C$ and $\mathcal{P}_{=1}(N) \subseteq C$: If not, we redefine the set C as follows: $C : C \cup \{N\} \cup \mathcal{P}_{=1}(N)$. Then, when we identify the sets N'_B and N'_W such that the theorem's requirements are satisfied for redefined C, clearly the theorem's requirements are also satisfied for C before we redefined it. That is why the assumption that $N \in C$ and $\mathcal{P}_{=1}(N) \subseteq C$ is innocuous.

We prove the theorem using mathematical induction on s as follows:

- **Base case**: There exist $N_B^1 \subseteq N'$ and $N_W^1 = N' \setminus N_B^1$ such that $|N_B^1| = k$, and for each $c \in C^1$, $||N_B^1 \cap c| - |N_W^1 \cap c|| \leq 1$.
- **Inductive Step:**

  (inductive hypothesis) For $s \in \{1, 2, 3, \cdots, t-1\}$, let $N_B^s \subseteq N'$ and $N_W^s = N' \setminus N_B^s$ be such that $|N_B^s| = k$, and for each $c \in C^1 \cup C^2 \cup \cdots \cup C^s$, $||N_B^s \cap c| - |N_W^s \cap c|| \leq 1$.

  (inductive conclusion) Then, there exist $N_B^{s+1} \subseteq N'$ and $N_W^{s+1} = N' \setminus N_B^{s+1}$ such that $\left|N_B^{s+1}\right| = k$, and for each $c \in C^1 \cup C^2 \cup \cdots \cup C^{s+1}$, $\left|\left|N_B^{s+1} \cap c\right| - \left|N_W^{s+1} \cap c\right|\right| \leq 1$.

Note that when the proof by mathematical induction is done, the sets $N'_B = N_B^t$ and $N'_W = N_W^t$ satisfy the requirements in the theorem.

Showing the base case is trivial: Clearly, $C^1 = \{N\}$. Then, any two sets $N_B^1 \subseteq N'$ and $N_W^1 = N' \setminus N_B^1$, where $|N_B^1| = k$, will be as required.

We now show the inductive step: Suppose the inductive hypothesis is true. Let $N_B^{s+1} = N_B^s$ and $N_W^{s+1} = N_W^s$. If for each $c \in C^{s+1}$, $\left|\left|N_B^{s+1} \cap c\right| - \left|N_W^{s+1} \cap c\right|\right| \leq 1$, we are done. Thus, suppose that for some $c \in C^{s+1}$, $\left|\left|N_B^{s+1} \cap c\right| - \left|N_W^{s+1} \cap c\right|\right| \geq 2$. Wlog, let $\left|N_B^{s+1} \cap c\right| - \left|N_W^{s+1} \cap c\right| \geq 2$. (The arguments are similar for the case when $\left|N_W^{s+1} \cap c\right| - \left|N_B^{s+1} \cap c\right| \geq 2$.) Let $c^* \in C^s$ be the mother of c. By the inductive hypothesis,

$$-1 \leq \left|N_B^{s+1} \cap c^*\right| - \left|N_W^{s+1} \cap c^*\right| = \sum_{l=1}^{\#(c^*)} \left(\left|N_B^{s+1} \cap c_l^*\right| - \left|N_W^{s+1} \cap c_l^*\right|\right) \leq 1.$$

Since c is a child of c^* and $\left|N_B^{s+1} \cap c\right| - \left|N_W^{s+1} \cap c\right| \geq 2$, the above inequality implies that there exists $l \in \{1, \cdots, \#(c^*)\}$ such that

$$\left|N_B^{s+1} \cap c_l^*\right| - \left|N_W^{s+1} \cap c_l^*\right| \leq -1.$$

Thus, $\left|N_W^{s+1} \cap c_l^*\right| \geq \left|N_B^{s+1} \cap c_l^*\right| + 1 \geq 1$. From above, we also know that $\left|N_B^{s+1} \cap c\right| \geq \left|N_W^{s+1} \cap c\right| + 2 \geq 2$. Thus, $N_W^{s+1} \cap c_l^* \neq \emptyset$ and $N_B^{s+1} \cap c \neq \emptyset$. Let $j^W \in N_W^{s+1} \cap c_l^*$ and $j^B \in N_B^{s+1} \cap c$. We redefine the sets N_B^{s+1} and N_W^{s+1} as follows:

$$N_B^{s+1} : N_B^{s+1} \setminus \left\{j^B\right\} \cup \left\{j^W\right\} \text{ and } N_W^{s+1} : N_W^{s+1} \setminus \left\{j^W\right\} \cup \left\{j^B\right\}.$$

Note that:

- for c_l^*, after N_B^{s+1} and N_W^{s+1} are redefined, the value $\left|\left|N_B^{s+1} \cap c_l^*\right| - \left|N_W^{s+1} \cap c_l^*\right|\right|$ becomes smaller or the same as before;

- for c, after N_B^{s+1} and N_W^{s+1} are redefined, the value $\left|\left|N_B^{s+1} \cap c\right| - \left|N_W^{s+1} \cap c\right|\right|$ becomes smaller;
- for $\widetilde{c} \in C^1 \cup C^2 \cup \cdots \cup C^{s+1} \setminus \{c, c_l^*\}$, after N_B^{s+1} and N_W^{s+1} are redefined, the value $\left|\left|N_B^{s+1} \cap \widetilde{c}\right| - \left|N_W^{s+1} \cap \widetilde{c}\right|\right|$ remains unchanged.

Obviously, the above process can be iterated and the sets N_B^{s+1} and N_W^{s+1} can be redefined until the inductive conclusion is satisfied. This concludes our proof. ∎

Note that our proof of Theorem 1 is constructive and hence provides an algorithm as to how to obtain the desired sets N'_B and N'_W.

5 Laminar Equilibrium in two-resource RSGs

This section presents Theorem 2, the main result of our paper. Before that, however, we characterize the sort of profitable deviations that may arise in a two-resource RSG in a Nash equilibrium when $T_1 = \{1, 2\}$.

LEMMA 1 *In a two-resource RSG, suppose that $T_1 = \{1, 2\}$ and $T_2 = \emptyset$. Let allocation a be a Nash equilibrium such that for resources i and i', $|a_i| = q_i$ and $|a_{i'}| = q_{i'} - 1$. Then, for $c \in \mathcal{P}_{\geq 1}(N)$, a is c-stable if and only if the conditions C1, C2, and C3 below are satisfied:*

***C1**. if $|a_{i'} \cap c| = 0$ then $|a_i \cap c| \leq 1$;*

***C2**. if $\beta_i = \beta_{i'}$ and $|a_{i'} \cap c| > 0$ then $|a_i \cap c| \leq |a_{i'} \cap c| + 1$;*

***C3**. if $\beta_i < \beta_{i'}$ and $|a_{i'} \cap c| > 0$ then $|a_i \cap c| \leq |a_{i'} \cap c|$.*

Proof. Let $T_1 = \{1, 2\}$ and $T_2 = \emptyset$. Let a be a Nash equilibrium such that $|a_i| = q_i$ and $|a_{i'}| = q_{i'} - 1$. We prove the two parts of the biconditional statement separately.

(only if)

By way of contradiction, suppose that a is c-stable but one of the conditions in the lemma is not satisfied.

If C1 is not satisfied, then $a_{i'} \cap c = \emptyset$ and $|a_i \cap c| \geq 2$. Consider an agent $j \in a_i \cap c$. Note that the set $(a_i \cap c) \setminus \{j\}$ is non-empty. Consider the deviation (c_1, c_2) such that $c_{i'} = \{j\}$ and $c_i = c \setminus \{j\}$. It is clear that (c_1, c_2) is a profitable deviation by coalition c at a, a contradiction. Therefore, if a is c-stable the condition C1 is satisfied.

If C2 is not satisfied, then $\beta_i = \beta_{i'}$, $|a_{i'} \cap c| > 0$, and $|a_i \cap c| \geq |a_{i'} \cap c| + 2$. Let $|a_{i'} \cap c| = k$ and $|a_i \cap c| = k + 2 + s$, where $k > 0$ and $s \geq 0$. Let $\widetilde{c} \subset a_i \cap c$ be such that $|\widetilde{c}| = k + 1$. Consider the deviation (c_1, c_2) such that $c_{i'} = \widetilde{c}$ and $c_i = c \setminus \widetilde{c}$. It is clear that (c_1, c_2) is a profitable deviation by coalition c at a, a contradiction. Therefore, if a is c-stable the condition C2 is satisfied.

If C3 is not satisfied, then $\beta_i < \beta_{i'}$, $|a_{i'} \cap c| > 0$, and $|a_i \cap c| > |a_{i'} \cap c|$. Let $|a_{i'} \cap c| = k$ and $|a_i \cap c| = k+1+s$, where $k > 0$ and $s \geq 0$. Let $\widetilde{c} \subseteq a_i \cap c$ be such that $|\widetilde{c}| = k+1$. Consider the deviation (c_1, c_2) such that $c_{i'} = \widetilde{c}$ and $c_i = c \setminus \widetilde{c}$. It is clear that (c_1, c_2) is a profitable deviation by coalition c at a, a contradiction. Therefore, if a is c-stable the condition C3 is satisfied.

Therefore, if a is c-stable, then the conditions given in the lemma are all satisfied.

(if)

By way of contradiction, suppose that for allocation a the conditions C1, C2, C3 are satisfied but a is not c-stable. Then there exists a profitable deviation (c_1, c_2) at allocation a. Let $a' = a \circ (c_1, c_2)$.

Suppose that for some resource $i'' \in \{i, i'\}$, $|a'_{i''}| > q_{i''}$. Then $|a'_{i''}| > |a_{i''}|$. Since $|a'_{i''}| = |a_{i''}| - |a_{i''} \cap c| + |c_{i''}|$, we must have $|c_{i''}| > |a_{i''} \cap c|$. Then $c_{i''} \neq \emptyset$. Let $j \in c_{i''}$. Since $|a'_{i''}| > q_i$, the cost that j incurs at a' is greater than u. But at a the cost that j incurs is less than or equal to u (because a is a Nash equilibrium; see Proposition 1). This contradicts that (c_1, c_2) is a profitable deviation at a. Thus, it must be that for each resource $i'' \in \{i, i'\}$, $a'_{i''} \leq q_{i''}$.

Note that $|N| = |a_i| + |a_{i'}| = q_i + q_{i'} - 1$. Then the fact that for each resource $i'' \in \{i, i'\}$, $a'_{i''} \leq q_{i''}$, implies that either $|a'_i| = q_i$ and $|a'_{i'}| = q_{i'} - 1$, or $|a'_i| = q_i - 1$ and $|a'_{i'}| = q_{i'}$.

Suppose that $|a'_i| = q_i$ and $|a'_{i'}| = q_{i'} - 1$. Then it is clear that the sum of the costs incurred by members of coalition c is the same at allocations a and a'. But then, if at a' an agent in c is better off (compared to at a), it must be that another agent in c is worse off at a'. But then (c_1, c_2) cannot be a profitable deviation at a, a contradiction. Therefore, we must have $|a'_i| = q_i - 1$ and $|a'_{i'}| = q_{i'}$.

If $a_{i'} \cap c = \emptyset$, then by C1 we get $|a_i \cap c| \leq 1$. Since c is non-empty, we get $|a_i \cap c| = 1$ and $|c| = 1$. But then (c_1, c_2) is a profitable deviation by a single-agent coalition, contradicting that a is a Nash equilibrium. Therefore, we obtain that $a_{i'} \cap c \neq \emptyset$. Let $|a_{i'} \cap c| = k$ where $k > 0$.

Suppose that $\beta_i > \beta_{i'}$. Consider an agent $j \in a_{i'} \cap c$. Note that the cost that j incurs at a is $\beta_{i'}$, and the cost that j incurs at a' is either β_i or u. Either way j is worse off at allocation a', contradicting that (c_1, c_2) is a profitable deviation at a. Therefore, $\beta_i \leq \beta_{i'}$.

Suppose that $\beta_i = \beta_{i'}$. Then, by C2, we find that $|a_i \cap c| = s$ where $s \leq k + 1$. Since $|a'_i| = q_i - 1$, $|a_i| = q_i$, and $|a'_i| = |a_i| - |a_i \cap c| + |a'_i \cap c|$, we obtain that $|a'_i \cap c| = |a_i \cap c| - 1$. Then $|a'_i \cap c| = s - 1$. At a the agents in $a_{i'} \cap c$ incur a cost equal to $\beta_{i'} < u$. Hence, at a' they cannot be assigned to resource i' (where the cost incurred is u). Therefore, $a_{i'} \cap c \subseteq a'_i \cap c$. Then $|a'_i \cap c| \geq |a_{i'} \cap c|$. Therefore, $s - 1 \geq k$. Since we also know that $s \leq k + 1$, we obtain that $s = k + 1$. Then $a_{i'} \cap c = a'_i \cap c$. This means that at a', agents in $a_{i'} \cap c$ are assigned to resource i and incur a cost equal to β_i, and agents in $a_i \cap c$ are assigned to resource i' and incur a cost equal to u. But then all agents are equally well off at a' and a, contradicting that (c_1, c_2) is a profitable deviation at a. Therefore, $\beta_i \neq \beta_{i'}$.

Suppose that $\beta_i < \beta_{i'}$. Then, by C3, we find that $|a_i \cap c| = s$ where $s \leq k$. Since $|a'_i| = q_i - 1$, $|a_i| = q_i$, and $|a'_i| = |a_i| - |a_i \cap c| + |a'_i \cap c|$, we obtain that $|a'_i \cap c| = |a_i \cap c| - 1$. Then $|a'_i \cap c| = s - 1$. Note that at a the agents in $a_{i'} \cap c$ incur a cost equal to $\beta_{i'} < u$, and hence at a' they cannot be assigned to resource i' (where the cost incurred is u). Therefore, $a_{i'} \cap c \subseteq a'_i \cap c$. Then $|a'_i \cap c| \geq |a_{i'} \cap c|$. Therefore, $s - 1 \geq k$. But this contradicts with the fact that $s \leq k$. Therefore, $\beta_i < \beta_{i'}$ cannot be true.

Since our supposition that (c_1, c_2) is a profitable deviation at a always leads to a contradiction, we find that when the conditions C1, C2, C3 are satisfied, the allocation a is c-stable.

This concludes our proof. ■

Before presenting and proving the main result of our paper, we will introduce some new tools.

The α-*value* of an allocation a w.r.t. a coalition structure C, to be denoted by $\alpha(a, C)$, is defined as follows:

$$\begin{aligned} \alpha(a, C) &= \sum_{i \in M} \sum_{c \in C} 1(c, a_i), \text{ where} \\ 1(c, a_i) &: \begin{cases} 1 \text{ if } c \cap a_i \neq \emptyset \\ 0 \text{ otherwise} \end{cases} . \end{aligned}$$

Loosely speaking, the α-*value* of allocation a is a cumulative measure of how "widely" coalitions are spread to resources at allocation a.

The β-*value* of an allocation a, to be denoted by $\beta(a)$, is defined as follows:

$$\beta(a) = \sum_{i \in M} f_i(|a_i|).$$

That is, the β-*value* of allocation a is the sum of the costs at resources at allocation a.

We say that allocation a' $\alpha\beta$-dominates allocation a w.r.t. C if $\alpha(a', C) > \alpha(a, C)$ or if $\alpha(a', C) = \alpha(a, C)$ and $\beta(a') < \beta(a)$.

Let $A \subseteq \mathcal{A}$ be a subset of allocations. Clearly, there exists $a \in A$ such that, for each $a' \in A \setminus \{a\}$, either a $\alpha\beta$-dominates a' w.r.t. C, or a and a' cannot be compared according to the $\alpha\beta$-domination relation w.r.t. C. We refer to such an allocation a as a "maximal element in A according to the $\alpha\beta$-domination relation w.r.t. C." Note that there may be more than one maximal elements in A.

We are now ready to present the main result of our paper.

THEOREM 2 *In a two-resource RSG, for any laminar coalition structure C, there exists a C-stable allocation.*

Proof. We show the existence of a C-stable allocation separately for the following three cases:

- Case 1: $T_2 \neq \emptyset$.
- Case 2: $T_1 = \{1, 2\}$ and $\beta_1 = \beta_2$.
- Case 3: $T_1 = \{1, 2\}$ and $\beta_1 \neq \beta_2$.

Let allocation a be a Nash equilibrium. (Its existence is by Proposition 1.)

Case 1: $T_2 \neq \emptyset$.
By Proposition 1, $|T_1| = |T_2| = 1$; and $|a_1| = q_1$ and $|a_2| = q_2$. But then, by Proposition 2, a is a super strong equilibrium and hence it is C-stable.

Case 2: $T_1 = \{1, 2\}$ and $\beta_1 = \beta_2$.
By Proposition 1, either $|a_1| = q_1 - 1$ and $|a_2| = q_2$ or $|a_1| = q_1$ and $|a_2| = q_2 - 1$. Thus, $|N| = q_1 + q_2 - 1$.

Let $k \geq 1$ be such that $|N| = 2k - 1$ or $|N| = 2k$. Wlog, let $q_1 \leq q_2$. Then, $q_1 \leq k \leq q_2$.

By the two-color theorem of laminarity, there exist $N'_B \subseteq N$ and $N'_W = N \setminus N'_B$ such that $|N'_B| = k$ and for each $c \in C$, $|N'_B \cap c| \leq |N'_W \cap c| + 1$.

Let $\widetilde{N}_B \subseteq N'_B$ be such that $\left|\widetilde{N}_B\right| = q_1$. Let $\widetilde{N}_W = N \setminus \widetilde{N}_B$. Note that $\left|\widetilde{N}_W\right| = q_2 - 1$. Since $\widetilde{N}_B \subseteq N'_B$ and $N'_W \subseteq \widetilde{N}_W$, we obtain that for each $c \in C$, $\left|\widetilde{N}_B \cap c\right| \leq \left|\widetilde{N}_W \cap c\right| + 1$. Therefore, for allocation a' such that $a'_1 = \widetilde{N}_B$ and $a'_2 = \widetilde{N}_W$, the conditions C1 and C2 in Lemma 1 are satisfied while the condition C3 is not applicable. (To ease comparison with lemma conditions, note that i and i' in the lemma statement are 1 and 2 in here, in order.) Therefore, by Lemma 1, a' is C-stable.

Case 3: $T_1 = \{1, 2\}$ and $\beta_1 \neq \beta_2$.

If a is C-stable, we are done. If not, we proceed as follows: We show the existence of an allocation a' such that a' is a Nash equilibrium and a' $\alpha\beta$-dominates a w.r.t. C. This proves that a C-stable allocation exists because: If a' turns out to be C-stable, we are done. Otherwise, we can iterate the same arguments: We can find an allocation a'' such that a'' is a Nash equilibrium and a'' $\alpha\beta$-dominates a' w.r.t. C, and so on. Since there exists a maximal element in the set of Nash equilibria according to the $\alpha\beta$-domination relation w.r.t. C, our iterations must eventually yield a C-stable allocation.

Therefore, suppose that a is not C-stable. Let $i, i' \in \{1, 2\}$ be such that $|a_i| = q_i$ and $|a_{i'}| = q_{i'} - 1$.

By Lemma 2, there exists $c \in C$ such that one of the conditions C1, C2, and C3 in Lemma 2 is not satisfied. Since $\beta_1 \neq \beta_2$, C2 is not applicable. Thus, either C1 or C3 is not satisfied.

Suppose that the condition C1 is not satisfied. Then, there exists $c \in C$ such that $|a_{i'} \cap c| = 0$ and $|a_i \cap c| \geq 2$. Let $j, j' \in a_i \cap c$, $j \neq j'$. Let allocation a' be such that $a'_i = a_i \setminus \{j'\}$ and $a'_{i'} = a_{i'} \cup \{j'\}$; hence, $j \in a'_i \cap c$, $j' \in a'_{i'} \cap c$, and $1(c, a'_i) = 1(c, a'_{i'}) = 1$. By Proposition 1, a' is a Nash equilibrium. Also, note that $\alpha(a', C) > \alpha(a, C)$ because:

- For each $c' \in C$ such that $c' \cap c = \emptyset$,

$$1(c', a_i) + 1(c', a_{i'}) = 1(c', a'_i) + 1(c', a'_{i'}).$$

  (Because agents in $N \setminus c$ are allocated to resources in exactly the same way at allocations a and a'.)

- For each $c' \in C$ such that $c \subset c'$,

$$1(c', a'_i) + 1(c', a'_{i'}) \geq 1(c', a_i) + 1(c', a_{i'}).$$

  (Because $1(c, a'_i) + 1(c, a'_{i'}) = 2$ and hence $1(c', a'_i) + 1(c', a'_{i'}) = 2$.)

- For c,

$$1(c, a'_i) + 1(c, a'_{i'}) > 1(c, a_i) + 1(c, a_{i'}).$$

  (Because $1(c, a'_i) + 1(c, a'_{i'}) = 2$ and $1(c, a_i) + 1(c, a_{i'}) = 1$.)

- For each $c' \in C$ such that $c' \subset c$,

$$1(c', a'_i) + 1(c', a'_{i'}) \geq 1(c', a_i) + 1(c', a_{i'}).$$

(Because $1(c, a_i) + 1(c, a_{i'}) = 1$, and hence, $1(c', a_i) + 1(c', a_{i'}) = 1$.)

Thus, as required, allocation a' is a Nash equilibrium and a' $\alpha\beta$-dominates a w.r.t. C.

Suppose that the condition C3 is not satisfied. Thus, $\beta_i < \beta_{i'}$ and there exists $c \in C$ such that $|a_i \cap c| > |a_{i'} \cap c| > 0$. Let k be such that $|a_{i'} \cap c| = k - 1$. Note that $|a_i \cap c| \geq k \geq 2$.

For each $j \in a_{i'} \cap c$, we define agent $\widetilde{j}$ as follows: Let $\widetilde{c} \in C$ be such that $\widetilde{c} \subseteq c$, $a_i \cap \widetilde{c} \neq \emptyset$, and there does not exist $\overline{c} \in C$ such that $\overline{c} \subset \widetilde{c}$ and $a_i \cap \overline{c} \neq \emptyset$. Let $\widetilde{j}$ be the smallest-index agent in $a_i \cap \widetilde{c}$.

Let $S_i = \left\{\widetilde{j} \,|\, j \in a_{i'} \cap c\right\}$. Note that, since $|a_{i'} \cap c| = k - 1$, $|S_i| \leq k - 1$. Let $\overline{S}_i$ be such that $S_i \subset \overline{S}_i \subseteq (a_i \cap c)$ and $\left|\overline{S}_i\right| = k$. Let $\overline{S}_{i,i'} = \overline{S}_i \cup (a_{i'} \cap c)$. Note that:

- $\left|\overline{S}_{i,i'}\right| = 2k - 1 \geq 3$.
- and for each $c' \in C$ such that $c' \subseteq c$ and $1(c', a_i) + 1(c', a_{i'}) = 2$, $\left|\overline{S}_{i,i'} \cap c'\right| \geq 2$. ♣

  (This is because of how we defined $\widetilde{j}$ and $\widetilde{c}$ above: for $j \in a_{i'} \cap c$, agent $\widetilde{j}$ is selected from within the set $a_i \cap \widetilde{c}$ where $\widetilde{c} \subseteq c'$; thus, $\widetilde{j} \neq j$ and $j, \widetilde{j} \in \overline{S}_{i,i'} \cap c'$.)

We now apply the two-color theorem of laminarity by setting $N' = \overline{S}_{i,i'}$: There exist $N'_B \subseteq \overline{S}_{i,i'}$ and $N'_W = \overline{S}_{i,i'} \setminus \overline{S}_{i,i'}$ such that $|N'_B| = k$ and for each $\widetilde{c} \in C$, $||N'_B \cap \widetilde{c}| - |N'_W \cap \widetilde{c}|| \leq 1$. Let a' be the allocation such that $a'_i = (a_i \setminus c) \cup N'_W$ and $a'_{i'} = (a_i \setminus c) \cup N'_B$. Clearly, at a' we have $|a'_i| = q_i - 1$ and $|a'_{i'}| = q_{i'}$. Hence, by Proposition 1, a' is a Nash equilibrium. Note that $\alpha(a', C) \geq \alpha(a, C)$ because:

- For each $c' \in C$ such that $c' \cap c = \emptyset$,

$$1(c', a_i) + 1(c', a_{i'}) = 1(c', a'_i) + 1(c', a'_{i'}).$$

(Because agents in $N \setminus c$ are allocated to resources in exactly the same way at allocations a and a'.)

- For each $c' \in C$ such that $c \subseteq c'$,

$$1(c', a'_i) + 1(c', a'_{i'}) \geq 1(c', a_i) + 1(c', a_{i'}).$$

(Because $\overline{S}_{i,i'} \subseteq c$, $\left|\overline{S}_{i,i'}\right| \geq 3$, and hence, by application of the two-color theorem of laminarity we obtain that $1\left(c', a_i'\right) = 1\left(c', a_{i'}'\right) = 1$.)

- For each $c' \in C$ such that $c' \subset c$ and $|c'| = 1$,

$$1\left(c', a_i'\right) + 1\left(c', a_{i'}'\right) = 1\left(c', a_i\right) + 1\left(c', a_{i'}\right) = 1.$$

(Because at any allocation a single agent is assigned to exactly one resource.)

- For each $c' \in C$ such that $c' \subset c$ and $|c'| \geq 2$,

$$1\left(c', a_i'\right) + 1\left(c', a_{i'}'\right) \geq 1\left(c', a_i\right) + 1\left(c', a_{i'}\right).$$

(Because: If $c' \subseteq (a_i \cap c)$ or $c' \subseteq (a_{i'} \cap c)$, we get $1\left(c', a_i\right) + 1\left(c', a_{i'}\right) = 1$ and the desired result follows. If $1(c', a_i) + 1(c', a_{i'}) = 2$, then $\left|\overline{S}_{i,i'} \cap c'\right| \geq 2$. (See the bullet argument above indicated with ♣.) Hence the desired result follows by application of the two-color theorem of laminarity.

Note that $\beta(a') = u + \beta_i$, $\beta(a) = u + \beta_{i'}$, and since $\beta_i < \beta_{i'}$, we get $\beta(a') < \beta(a)$. Since $\alpha\left(a', C\right) \geq \alpha\left(a, C\right)$ and $\beta(a') < \beta(a)$, we obtain that a' $\alpha\beta$-dominates a w.r.t. C. Thus, as required, allocation a' is a Nash equilibrium and a' $\alpha\beta$-dominates a w.r.t. C.

This concludes our proof. ■

Note that our proof of Theorem 2 is constructive and hence provides an algorithm as to how to obtain in a two-resource RSG a C-stable allocation, for C laminar.

References

[1] Anshelevich, Elliot, Bugra Caskurlu, and Ameya Hate. "Partition equilibrium always exists in resource selection games." *Theory of Computing Systems* 53, no. 1 (2013): 73-85.

[2] Ashlagi, Itai, Piotr Krysta, and Moshe Tennenholtz. "Social context games." In *International Workshop on Internet and Network Economics*, pp. 675-683. Springer, Berlin, Heidelberg, 2008.

[3] Aumann, Robert. "Acceptable Points in General Cooperative n-Person Games." *Contributions to the Theory of Games* 4 (1959).

[4] Feldman, Michal, and Moshe Tennenholtz. “Structured coalitions in resource selection games.” *ACM Transactions on Intelligent Systems and Technology (TIST)* 1, no. 1 (2010): 4.

[5] Hayrapetyan, Ara, Éva Tardos, and Tom Wexler. “The effect of collusion in congestion games.” In *Proceedings of the thirty-eighth annual ACM symposium on Theory of computing*, pp. 89-98. ACM, 2006.

[6] Hoefer, Martin, Michal Penn, Maria Polukarov, Alexander Skopalik, and Berthold Vöcking. “Considerate equilibrium.” In *IJCAI*, pp. 234-239. 2011.

[7] Holzman, Ron, and Nissan Law-Yone. “Strong equilibrium in congestion games.” *Games and economic behavior* 21, no. 1-2 (1997): 85-101.

[8] Konishi, Hideo, Michel Le Breton, and Shlomo Weber. “Equilibria in a model with partial rivalry.” *Journal of Economic Theory* 72, no. 1 (1997): 225-237.

[9] Kuniavsky, Sergey, and Rann Smorodinsky. “Equilibrium and potential in coalitional congestion games.” *Theory and decision* 76, no. 1 (2014): 69-79.

[10] Milinski, Manfred. “An evolutionarily stable feeding strategy in sticklebacks.” *Ethology* 51, no. 1 (1979): 36-40.

[11] Milchtaich, Igal. “Congestion games with player-specific payoff functions.” *Games and economic behavior* 13, no. 1 (1996): 111-124.

[12] Monderer, Dov, and Lloyd S. Shapley. “Potential games.” *Games and economic behavior* 14.1 (1996): 124-143.

[13] Quint, Thomas, and Martin Shubik. A model of migration. Vol. 1088. Cowles Foundation for Research in Economics, 1994.

[14] Rosenthal, Robert W. “A class of games possessing pure-strategy Nash equilibria.” *International Journal of Game Theory* 2, no. 1(1973): 65-67.